Brownian motion and Stochastic Calculus
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Assignment 10—solutions

Exercise 1

We fix a standard one-dimensional $(\mathbb{F},\mathbb{P})$–Brownian motion.

1) Show that for any $C^{1,2}$ function $f:[0,+\infty)\times\mathbb{R}\longrightarrow\mathbb{R}$, such that there exists some continuous function $C:[0,+\infty)\longrightarrow[0,+\infty)$ with
$$|\partial_x f(t,x)|\leq C(t)\mathrm{e}^{C(t)|x|},\ (t,x)\in[0,+\infty)\times\mathbb{R}, \tag{0.1}$$
the process $(f(t,B_t))_{t\geq 0}$ will be an $(\mathbb{F},\mathbb{P})$-martingale if and only if
$$\partial_t f(t,x)+\frac{1}{2}\partial^2_{xx}f(t,x)=0,\ (t,x)\in[0,+\infty)\times\mathbb{R}. \tag{0.2}$$

2) in this question, we are looking for functions f of the form
$$f(t,x)=\sum_{i=0}^n\sum_{j=0}^n a_{i,j}t^ix^j,\ (t,x)\in[0,+\infty)\times\mathbb{R},$$
for some integer n and real numbers $(a_{i,j})_{(i,j)\in\{0,\dots,n\}^2}$. Show that the process $f(t,B_t)$ is an $(\mathbb{F},\mathbb{P})$-martingale if and only if the $(a_{0,j})_{j\in\{0,\dots,n\}}$ are arbitrarily fixed and
$$\begin{cases}a_{i,j}=(-1)^i\dfrac{(j+2i)!}{2i!j!}a_{0,j+2i},\ j+2i\leq n,\\ a_{i,j}=0,\ j+2i>n,\end{cases}$$

1) **Indeed, by Itô's formula the Itô process $(f(t,B_t))_{t\geq0}$ is an $(\mathbb{F}^{B,\mathbb{P}},\mathbb{P})$–local martingale if and only if its drift is equal to 0 with $\mathbb{P}$-probability 1, that is**
$$\partial_t f(t,B_t)+\frac{1}{2}\partial^2_{xx}f(t,B_t)=0,\ t\geq0,\ \mathbb{P}\text{–a.s.}$$
Since the support of the $\mathbb{P}$-distribution of B_t is $\mathbb{R}$, we deduce the desired condition. Next, since inequality (0.1) holds, the volatility of $(f(t,B_t))_{t\geq0}$ is automatically in $\mathbb{H}^2(\mathbb{R},\mathbb{F}^{B,\mathbb{P}},\mathbb{P})$, which shows the martingale property.

2) **In this case, inequality (0.1) is obviously satisfied, and direct computations prove that (0.2) holds if and only if $a_{1,0}=a_{1,1}=0$ when $n=1$ (the case $n=0$ is trivial), and when $n\geq2$**
$$\begin{cases}a_{i+1,j}=-\dfrac{(j+2)(j+1)}{2(i+1)}a_{i,j+2},\ i\in\{0,\dots,n-1\},\ j\in\{0,\dots,n-2\},\\ a_{n,j+2}=0,\ j\in\{0,\dots,n-2\},\\ a_{i+1,n-1}=0,\ i\in\{0,\dots,n-1\},\\ a_{i+1,n}=0,\ i\in\{0,\dots,n-1\}.\end{cases}$$

It can then easily be checked that this equivalent to having the $(a_{0,j})_{j\in\{0,\dots,n\}}$ arbitrarily fixed and
$$\begin{cases}a_{i,j}=(-1)^i\dfrac{(j+2i)!}{2i!j!}a_{0,j+2i},\ j+2i\leq n,\\ a_{i,j}=0,\ j+2i>n,\end{cases}$$

Exercise 2

Consider, for any $x \in \mathbb{R}^d$, the SDE

$$\mathrm{d}X_t^x = a(X_t^x)\mathrm{d}t + b(X_t^x)\mathrm{d}W_t,\ X_0^x = x,$$

where W is a $\mathbb{R}^m$-valued Brownian motion, $a : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $b : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times m}$ are measurable and locally bounded. We fix a non-empty, bounded open subset U of $\mathbb{R}^d$ and assume that for any $x \in U$, we have with $T_U^x := \inf\{s \geq 0 : X_s^x \notin U\}$, that T_U^x is $\mathbb{P}$-integrable.

Moreover, consider the boundary problem

$$Lu(x) + c(x)u(x) = -f(x),\ \text{for}\ x \in U,\ u(x) = g(x),\ \text{for}\ x \in \partial U,$$

where $f \in C_b(U)$, $g \in C_b(\partial U)$, $c \leq 0$ is a uniformly bounded function on $\mathbb{R}^d$, and L is defined by

$$Lf(x) := \sum_{i=1}^d a^i(x)\frac{\partial f}{\partial x^i}(x) + \frac{1}{2}\sum_{(i,j)\in\{1,\dots,d\}^2} \left(bb^\top\right)^{ij}(x)\,\frac{\partial^2 f}{\partial x^i\,\partial x^j}(x).$$

Show that if $u \in C^2(U) \cap C(\bar{U})$ is a solution of the above boundary problem and $(X_t^x)_{t\geq 0}$ is a solution of the SDE for some $x \in U$, then

$$u(x) = \mathbb{E}^{\mathbb{P}}\left[g(X^x_{T_U^x})\exp\left(\int_0^{T_U^x} c(X_s^x)\mathrm{d}s\right)\right] + \mathbb{E}^{\mathbb{P}}\left[\int_0^{T_U^x} f(X_s^x)\exp\left(\int_0^s c(X_r^x)\mathrm{d}r\right)\mathrm{d}s\right].$$

For $m \in \mathbb{N}^\star$ large enough so that $\frac{1}{m} < d(x, U^c)$, we define

$$T_m := \inf\left\{s \geq 0 : d(X_s^x, U^c) \leq 1/m\right\},$$

and construct $u_m \in C_c^2(\mathbb{R}^d, \mathbb{R})$ such that $u = u_m$ on $\left\{z \in U : d(z, U^c) \geq 1/m\right\}$. We apply Itô's formula to $u_m(X_t^x)\exp\left(\int_0^t c(X_s^x)\mathrm{d}s\right)$, take then the expectation and use that the local martingale is a true martingale as b is locally bounded and $u \in C_c^2$ to obtain that

$$\mathbb{E}^{\mathbb{P}}\left[u_m(X^x_{t\wedge T_m^x})\exp\left(\int_0^{t\wedge T_m^x} c(X_s^x)\mathrm{d}s\right)\right] - u_m(x) = \mathbb{E}^{\mathbb{P}}\left[\int_0^{t\wedge T_m^x}\left(Lu_m(X_s^x) + c(X_s^x)u_m(X_s^x)\right)\exp\left(\int_0^s c(X_r^x)\mathrm{d}r\right)\mathrm{d}s\right].$$

Now, as $u_m = u$ on $\left\{z \in U : d(z, U^c) \geq \frac{1}{m}\right\}$, by definition of T_m^x and as u is the solution of the boundary problem, we obtain that

$$u(x) = \mathbb{E}\mathbb{P}\left[u(X^x_{t\wedge T_m^x})\exp\left(\int_0^{t\wedge T_m^x} c(X_s^x)\mathrm{d}s\right)\right] + \mathbb{E}^{\mathbb{P}}\left[\int_0^{t\wedge T_m^x} f(X_s^x)\exp\left(\int_0^s c(X_r^x)\mathrm{d}r\right)\mathrm{d}s\right].$$

Since $T_m^x \uparrow T_U^x < \infty$, we can let $t \to \infty$ and then $m \to \infty$ to conclude, by the dominated convergence theorem, that

$$u(x) = \mathbb{E}^{\mathbb{P}}\left[g(X^x_{T_U^x})\exp\left(\int_0^{T_U^x} c(X_s^x)\mathrm{d}s\right)\right] + \mathbb{E}^{\mathbb{P}}\left[\int_0^{T_U^x} f(X_s^x)\exp\left(\int_0^s c(X_r^x)\mathrm{d}r\right)\mathrm{d}s\right].$$

Exercise 3

Let $(B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion.

1) Show that the SDE

$$X_t = x + \int_0^t \sqrt{1 + X_s^2}\,\mathrm{d}B_s + \frac{1}{2}\int_0^t X_s\mathrm{d}s, \tag{0.3}$$

admits a unique strong solution for all $x \in \mathbb{R}$.

2) Fix $x \in \mathbb{R}$ and $(\beta_t, \gamma_t)_{t\geq 0}$ two independent one-dimensional Brownian motions. Show that

$$Y_t := \exp(\beta_t)\bigg(x + \int_0^t \exp(-\beta_s)\mathrm{d}\gamma_s\bigg),\ t \geq 0,$$

is well-defined and solves (0.3) for some well-chosen Brownian motion B. Deduce that for $a := \operatorname{argsinh}(x)$,

$$(Y_t,\ t \geq 0) \overset{\text{(law)}}{=} (\sinh(a + B_t),\ t \geq 0).$$

3) We now go to a slightly more general setting.

a) Show that if the map $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^2 diffeomorphism from $\mathbb{R}$, then $\Phi_t := \varphi(B_t)$ satisfies

$$\Phi_t = \varphi(0) + \int_0^t \sigma(\Phi_s)\mathrm{d}B_s + \int_0^t b(\Phi_s)\mathrm{d}s, \tag{0.4}$$

where

$$\sigma(x) := (\varphi' \circ \varphi^{(-1)})(x),\ b(x) := \frac{1}{2}(\varphi'' \circ \varphi^{(-1)})(x).$$

b) Conversely, if $\sigma, b : \mathbb{R} \longrightarrow \mathbb{R}$ are Lipschitz functions with appropriate growth, we know that the SDE (0.4) admits a unique strong solution. Under which conditions on (σ, b) can we solve the system

$$\varphi'(y) = \sigma(\varphi(y)),\ \varphi''(y) = 2b(\varphi(y)),$$

so that the solution of (0.4) is $\Phi_t = \varphi(B_t)$?

1) **The drift and volatility are clearly Lipschitz-continuous with linear growth, hence the standard Cauchy–Lipschitz theorem applies.**

2) **We have for any $t \geq 0$**

$$\mathbb{E}^{\mathbb{P}}\bigg[\int_0^t \mathrm{e}^{-2\beta_s}\mathrm{d}s\bigg] = \int_0^t \mathrm{e}^{2s}\mathrm{d}s < +\infty,$$

ensuring that Y is well-defined. Next, applying Itô's formula to Y, we get

$$\mathrm{d}Y_t = \bigg(x + \int_0^t \exp(-\beta_s)\mathrm{d}\gamma_s\bigg)\mathrm{e}^{\beta_t}\bigg(\mathrm{d}\beta_t + \frac{1}{2}\mathrm{d}t\bigg) + \mathrm{d}\gamma_t = \frac{1}{2}Y_t\mathrm{d}t + \sqrt{1+Y_t^2}\bigg(\frac{Y_t}{\sqrt{1+Y_t^2}}\mathrm{d}\beta_t + \frac{1}{\sqrt{1+Y_t^2}}\mathrm{d}\gamma_t\bigg).$$

Now, let $B := \int_0^\cdot \big(\frac{Y_t}{\sqrt{1+Y_t^2}}\mathrm{d}\beta_t + \frac{1}{\sqrt{1+Y_t^2}}\mathrm{d}\gamma_t\big)$. We have

$$[B]_t = t,$$

ensuring by Lévy's characterisation that B is a Brownian motion.

Next, it is direct to check that $X_t := (\sinh(a + B_t)$ is the strong solution to the SDE, which gives the desired result by uniqueness in law.

3)*a)* **It is a simple application of Itô's formula. Indeed, we have**

$$\Phi_t = \Phi_0 + \int_0^t \varphi'(B_s)\mathrm{d}B_s + \frac{1}{2}\int_0^t \varphi''(B_s)\mathrm{d}s,$$

and it suffices to notice that $B_t = \varphi^{(-1)}(\Phi_t)$, $t \geq 0$.

b) **If we can find φ as a C^2 diffeomorphism satisfying the two ODEs, then clearly $\Phi = \varphi(B)$. Now, it is necessary for this that**

$$\varphi'(y)\sigma'(\varphi(y)) = \sigma(\varphi(y))\sigma'(\varphi(y)) = 2b(\varphi(y)).$$

Hence, φ being a diffeomorphism on $\mathbb{R}$, this means that we must have

$$\sigma\sigma' = 2b.$$

Under this assumption, and if for instance $\sigma\sigma'$ is Lipschitz-continuous, and σ has a fixed sign, the result will hold.